

ON THE SINGULAR SETS OF A MODULE II

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Throughout this note, A and B will denote a (non-trivial) commutative Noetherian ring with a multiplicative identity element and M will denote a non-zero finitely generated A -module.

For every non-negative integer k , the set $S_k^*(M) = \{p \in \text{Spec}(A) \mid \text{depth } M_p + \dim A/p \leq k\}$ is called the **singular set of M** with respect to k .

It is known that when the ring A is homomorphic image of a biequidimensional regular ring, then the singular sets of M are all closed in the Zariski topology on $\text{Spec}(A)$ (see [3; ch. IV, 5]).

A development of this famous theorem has been recently shown in the sense that if A is a homomorphic image of a biequidimensional Gorenstein ring, the singular sets of M are still closed (See [2]).

The purpose of this article is to show that if B is a homomorphic image of a Cohen-Macaulay local ring and N is a finitely generated B -module, then $S_k^*(N)$ is closed, for every finitely generated B -module N .

First we prove some preliminary lemmas and propositions which help us to conclude the subsequent main theorem. From now on, A will denote a Cohen-Macaulay local ring with the unique maximal ideal m , and $\hat{A}$ (respectively $\hat{M}$) will denote the completion of A (respectively M).

1. Proposition. Let $\phi : A \rightarrow \hat{A}$ be the natural homomorphism. Then for every $q \in \text{Spec}(\hat{A})$, $S_k^*(\hat{M}) \cap q^c = S_k^*(M) \cap q^c$ (for any ideal J of A we write J^c for $\phi^{-1}(J)$).

Proof. By [5; 23.3], $\text{depth}_{A_q}(M_p \otimes_{A_p} \hat{A}_q) = \text{depth}_{\hat{A}_q}(\hat{M}_q)$.

$\text{depth}_{\Lambda_p}(M_p) + \text{depth}(\hat{A}_q/pA_p\hat{A}_q)$, since.

$$\begin{aligned} \bar{\varphi} : A_p &\rightarrow \hat{A}_q \\ \frac{a}{s} &\rightarrow \frac{\varphi(a)}{\varphi(s)} \end{aligned}$$

is a flat homomorphism. Also we have

$$\begin{aligned} M_p \otimes_{\Lambda_p} \hat{A}_q &\cong (M \otimes_{\Lambda} A_p) \otimes_{\Lambda_p} \hat{A}_q \cong M \otimes_{\Lambda} (A_p \otimes_{\Lambda_p} \hat{A}_q) \\ &\cong M \otimes_{\Lambda} \hat{A}_q \cong M \otimes_{\Lambda} (\hat{A} \otimes_{\Lambda} \hat{A}_q) \cong (M \otimes_{\Lambda} \hat{A}) \otimes_{\Lambda} \hat{A}_q \\ &\cong \hat{M} \otimes_{\Lambda} \hat{A}_q \cong \hat{M}_q. \end{aligned}$$

Thus we conclude that

$$\text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) = \text{depth}_{\Lambda_p} M_p + \text{depth}(\hat{A}_q/pA_p\hat{A}_q).$$

On the other hand, since A is Cohen-Macaulay, $\hat{A}$ is a Cohen-Macaulay local ring; whence, by corollary of [5;23.3], $\hat{A}_q/pA_p\hat{A}_q$ is a Cohen-Macaulay ring. But

$$\hat{A}_q/pA_p\hat{A}_q = \hat{A}_q/p\hat{A}_q.$$

Hence

$$\text{depth}\left(\frac{\hat{A}_q}{pA_p\hat{A}_q}\right) = \dim\left(\frac{\hat{A}_q}{p\hat{A}_q}\right).$$

Moreover, by [5;15.1],

$$\text{ht } q = \text{ht } p + \dim\left(\frac{\hat{A}_q}{p\hat{A}_q}\right).$$

Hence

$$\text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) = \text{depth}_{\Lambda_p}(M_p) + \text{ht}_q - \text{ht}_p.$$

From which we get, by [5;17.4],

$$\begin{aligned} \text{depth}_{\hat{\Lambda}_q}(\hat{M}_q) + \dim\left(\frac{\hat{A}}{q}\right) &= \text{depth}_{\Lambda_p}(M_p) + \dim \hat{A} - \text{ht } p \\ &= \text{depth}_{\Lambda_p}(M_p) + \dim A - \text{ht } p \\ &= \text{depth}_{\Lambda_p}(M_p) + \dim\left(\frac{A}{q}\right) \end{aligned}$$

The result now follows.

2. Proposition. With the same assumption as in Proposition 1. Let $p, \hat{p} \in \text{Spec}(A)$ be prime ideals such that $p \subseteq \hat{p}$ and $p \in S_k^*(M)$. Then $\hat{p} \in S_k^*(M)$.

Proof. Since $\varphi: A \rightarrow \hat{A}$ is a faithfully flat homomorphism, there exists $\hat{q} \in \text{Spec}(\hat{A})$ for which $(\hat{q})^c = \hat{p}$ (by [5;7.3]). But φ has the going down property (see[5;9.5]). Hence there is a prime ideal

$q \in \text{Spec}(\hat{A})$ such that $q^c = p$ and $q \subseteq \hat{q}$. By Proposition 1, this implies that $q \in S_k^*(\hat{M})$. But $\hat{A}$ is a homomorphic image of a regular local ring (see[5;29.4(ii)]); thus by [3], $S_k^*(\hat{M})$ is a closed subset of $\text{Spec}(\hat{A})$ (note that, every Cohen-Macaulay local ring is biequidimensional ring). This implies that $\hat{q} \in S_k^*(\hat{M})$. Again from Proposition 1, this in turn implies that $(\hat{q})^c = \hat{p} \in S_k^*(M)$ as required.

3. Lemma. (See[4;ch.1, §6, Ex. 1]) Let $R \subseteq T$ be rings and p a minimal prime ideal in R . Then there exists in T a prime ideal contracting to p .

Proof. Let p be a minimal prime ideal of R . Set $S = R/p$, and

$$K = \{a \mid a \cap S = \phi \text{ \& } a \text{ is an ideal of } T\}.$$

Then K have a maximal element which is prime ideal of T . Let q be such prime ideal. Since $(q \cap R) \cap S = \phi$, we have $(q \cap R) \subseteq p$ and consequently $q \cap R = p$.

We now turn to the main theorem of the note.

4. Theorem. For every positive integer k , $S_k^*(M)$ is a closed subset of $\text{Spec}(A)$.

Proof: Since $S_k^*(\hat{M})$ is closed in $\text{Spec}(\hat{A})$, there exists an ideal J of $\hat{A}$ such that $V(J) = S_k^*(\hat{M})$. It is enough to show that

$$V(J^c) = S_k^*(M)$$

Let $p \in S_k^*(M)$. Hence there is $q \in S_k^*(\hat{A})$ such that $q^c = p$. Hence $q \in S_k^*(\hat{M})$. Thus $J \subseteq q$; this implies that $J^c \subseteq q^c = p$; i. e., $p \in V(J^c)$.

Now let $\hat{p} \in V(J^c)$. φ induces the one-to-one homomorphism

$$\tilde{\varphi} : A/J^c \rightarrow \hat{A}/J$$

$$a + J^c \rightarrow \varphi(a) + J.$$

There is also a minimal prime ideal of J^c as $\mathfrak{p}$ such that

$$J^c \subseteq \mathfrak{p} \subseteq \hat{\mathfrak{p}}.$$

Now by Lemma 3, there is q/J in $\text{Spec}(\hat{A}/J)$ such that

$$\tilde{\varphi}^{-1}(q/J) = \mathfrak{p}/J^c$$

Hence $\mathfrak{p} = q^c$ and $J \subseteq q$. Hence $q \in V(J) = S_k^*(\hat{M})$. It follows from Proposition 1 that $\hat{\mathfrak{p}} \in S_k^*(M)$. By Proposition 2, we conclude that $\hat{\mathfrak{p}} \in S_k^*(M)$.

Hence $V(J^c) = S_k^*(M)$ and $S_k^*(\hat{M})$ is closed as claimed.

5. Corollary. Let B be a homomorphic image of A . Then for every finitely generated B -module N the singular sets $S_k^*(N)$ are closed.

Proof. Let $f : A \rightarrow B$ be the relevant ring epimorphism. By [1;5], for every non-negative integer k ,

$$S_k^*(N) = \{p \in \text{Spec}(B) : f^{-1}(p) \in S_k^*(N|_A)\}$$

in which $N|_A$ is the module N to be considered by restriction of scalars by means of f . Since $S_k^*(N|_A)$ is a closed subset of $\text{Spec}(A)$, and $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a continuous map, we conclude that $f^{*-1}S_k^*(N|_A) = S_k^*(N)$ is a closed subset of $\text{Spec}(B)$.

References

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